Set Theory Theorems and Definitions

Set Membership, Equality, and Subsets

An element of a set is an object directly contained within that set. For example, $1 \in \{1, 2, 3\}$ and $\emptyset \in \{\emptyset\}$, but $1 \notin \emptyset$ and $1 \notin \{2, 3\}$. Note that $1 \notin \{\{1\}\}$, and $\{1\} \notin \{1\}$, but $\{1\} \in \{\{1\}\}$.

Two sets are equal if they contain the same elements. For example, we have that $\{1, 2\} = \{2, 1\}$ and that $\{\emptyset\} = \{\emptyset\}$. However, $\{\emptyset\} \neq \{\{\emptyset\}\}$, because each set contains an element the other does not. A set and a non-set are never equal; in particular, this means $x \neq \{x\}$ for any x.

A set A is a subset of a set B (denoted $A \subseteq B$) if every element of A is also an element of B:

$$\mathbb{N} \subseteq \mathbb{Z} \qquad \{1, 2, 3\} \subseteq \{1, 2, 3, 4\} \qquad \{1\} \subseteq \{1, \{1\}, \{\{1\}\}\}$$

Set Operations

The set $\{\ x \mid \textit{some property of } x\ \}$ is the set of all objects x that satisfy the given property. Formally, we have that $w \in \{\ x \mid \textit{some property of } x\ \}$ if and only if the specified property holds for w.

The set $A \cup B$ is the set $\{\ x \mid x \in A \text{ or } x \in B\ \}$. Equivalently, $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.

The set $A \cap B$ is the set $\{\ x \mid x \in A \text{ and } x \in B\ \}$. Equivalently, $x \in A \cap B$ precisely if $x \in A$ and $x \in B$.

The set $A - B$ is the set $\{\ x \mid x \in A \text{ and } x \notin B\ \}$. This set is also sometimes denoted $A \setminus B$.

The set $A \Delta B$ is the set $\{\ x \mid \text{exactly one of } x \in A \text{ and } x \in B \text{ is true}\ \}$.

Power Sets

The power set of a set S, denoted $\wp(S)$, is the set of all subsets of S. Using set-builder notation, this is the set $\wp(S) = \{\ T \mid T \subseteq S\ \}$. Cantor's Theorem states that $|S| < |\wp(S)|$ for every set S.

Special Sets

The set $\emptyset = \{\ \}$ is the empty set containing no elements.

The set $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$ is the set of all natural numbers. We treat 0 as a natural number.

The set $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is the set of all integers.

The set $\mathbb{R}$ consists of all the real numbers. The set $\mathbb{Q}$ consists of all rational numbers.

Cardinality

The cardinality of a finite set S (denoted $|S|$) is the natural number equal to the number of elements in that set. The cardinality of $\mathbb{N}$ (denoted $|\mathbb{N}|$) is $\aleph_0$ (pronounced "aleph-nought"). Two sets have the same cardinality if there is a way of pairing up each element of the two sets such that every element of each set is paired with exactly one element of the other set.

Important Theorems

Here are some useful theorems about sets that you're welcome to use however you see fit. We proved some of these in class, while others would make for great exercises.

Theorem: If A, B, and C are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

This theorem says that the $\subseteq$ relation is ***transitive***.

Theorem: If A and B are sets where $A \subseteq B$ and $B \subseteq A$, then $A = B$.

This is an ***extremely*** useful way to prove two sets are equal. In fact, if you ever find yourself needing to prove that two sets are equal, you might want to pull out this theorem!

Theorem: If S is a set, then $|S| < |\wp(S)|$.

This is ***Cantor's theorem***. We'll write a rigorous proof of in the third week of class.